

Computing the Distribution of the Maximum of a Gaussian Process

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Abstract

This paper deals with the problem of obtaining methods to compute the distribution of the maximum of a one-parameter stochastic process on a fixed interval, mainly in the Gaussian case. The main point is the relationship between the values of the maximum and crossings of the paths, via the so-called Rice's formulae for the factorial moments of crossings.

In certain relevant situations the formulae are adapted to the numerical computation of the distribution of the maximum and are more efficient than other numerical methods.

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Short Title: Distribution of the Maximum.

1 Introduction

Let $X = \{X_t : t \in \mathbb{R}\}$ be a stochastic process with real values and continuous paths defined on a probability space $(\Omega, \mathfrak{S}, P)$ and $M_T := \max\{X_t : t \in [0, T]\}$.

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The computation of the distribution function of the random variable $M_T F(T, u) := P(M_T \leq u), u \in \mathbb{R}$ by means of a closed formula based upon natural parameters of the process X is known only for a very restricted number of stochastic processes (and trivial functions of them): the Brownian Motion $\{W_t : t \geq 0\}$; the Brownian Bridge, $B_t := W_t - tW_1$ ($0 \leq t \leq 1$); $B_t - \int_0^1 B_s ds$ (Darling, 1983); the Brownian Motion with a linear drift (Shepp, 1979); $\int_0^t W_s ds + yt$ (McKean, 1963, Goldman, 1971 Lachal, 1991); the stationary Gaussian processes with covariance equal to:

1. $r(t) = e^{-|t|}$ (Ornstein-Uhlenbeck process, DeLong, 1981),
2. $r(t) = (1 - |t|)^+$, T a positive integer (Slepian process, Slepian 1961, Shepp, 1971),
3. $r(t)$ even, periodic with with period 2, $r(t) = 1 - \alpha|t|$ for $0 \leq |t| \leq 1, 0 < \alpha \leq 2$, (Shepp and Slepian 1976),
4. $r(t) = 1 - |t|/1 - \beta \vee -\beta/1 - \beta, |t| < 1 - \beta/\beta, 0 < \beta \leq 1/2, T = (1 - \beta)/\beta$ (Cressie 1980),
5. $r(t) = \cos t$.

Given the interest in $F(T, u)$ for a large diversity of theoretical and technical purposes an extensive literature has been developed towards:

1. Obtaining inequalities for $F(T, u)$, : Slepian (1962); Landau & Shepp (1970); Marcus & Shepp (1972); Fernique (1974); Borell (1975); Talagrand (1996) and references therein. A general review of classical results is in Adler (1990, 2000).
2. Describing the behaviour of $F(T, u)$ under various asymptotics : Qualls and Watanabe (1973); Piterbarg (1981, 1996); Leadbetter, Lingren and Rootzén (1983); Sun (1993); Berman (1985a, b, 1992); Berman & Kôno (1992) ; Talagrand (1988).
3. Studying the regularity of the distribution of M_T : Ylvisaker (1968); Tsirelson (1975); Weber (1985); Lifshitz (1995); Diebolt and Posse (1996), Azaïs and Wschebor (2000) and references therein.

Generally speaking, even though important results are associated with problems 1) 2) and 3) they only give limited answers to the computation of $F(T, u)$ for *fixed* T and u . As a consequence, Monte-Carlo methods based on the simulation of the paths of the continuous parameter process X are widely used, even though they are poor for theoretical purposes, expensive from the point of view of the number of elementary computations needed to assure that the error is below a given bound, and always depend on the quality of the random number generator being employed.

The approach in this paper is based upon some exact series that express $F(T, u)$ in terms of the factorial moments of the number of upcrossings. The underlying ideas have been known since a long time (Miroshin (1974)) especially combinatorial lemma 3.2 of which we give an independent proof. What is new is that we have been able to prove the convergence of the series in a general framework, instead of considering only some particular processes, thus providing a method that can be widely applied. On the other hand, even though Theorems 3.1 and 3.4 below do not refer specifically to Gaussian processes, at present we are able to apply them to the numerical computation of $F(T, u)$ only in Gaussian cases.

We have included a comparison between the complexities of the computations of $F(T, u)$ using the Rice series versus Monte-Carlo method for the relevant case of a general class of stationary Gaussian processes. It shows that the use of Rice series is a priori better. More important, it is self-controlling for the numerical errors. This implies that the a posteriori number of computations may be much smaller than the one required by simulation. In fact, in relevant cases, the actual computation is performed with a few terms in the Rice series. As examples, we have included tables for $F(T, u)$ for a number of Gaussian processes. We also pay some attention to certain problems arising naturally in performing the numerical computation of the moments of upcrossings that requires the use of some practical tools to be efficient.

We use the same method for stationary Gaussian processes verifying certain regularity conditions, to obtain upper and lower bounds for $F(T, u)$ that are sharp as $u \rightarrow +\infty$ but do not improve known results (Piterbarg, 1981). Recent improvements for this asymptotic behaviour (Bardet and Wschebor, 2000) are based on the study of the density of M_T (Azais and Wschebor, 2000).

The Note (Azais & Wschebor 1997) contains a part of the results in the present paper, without proofs.

Notations

Let $f : I \rightarrow \mathbb{R}$ be a function defined on the interval I of the real numbers,

$$C_u(f; I) := \{t \in I : f(t) = u\}$$

$$N_u(f; I) := \#(C_u(f; I))$$

denote respectively the set of roots of the equation $f(t) = u$ on the interval I and the number of these roots, with the convention $N_u(f; I) = +\infty$ if the set C_u is infinite. $N_u(f; I)$ is called the number of “crossings” of f with the “level” u on the interval I .

In the same way, if f is a differentiable function the number of “upcrossings” of f is defined by means of

$$U_u(f; I) := \#(\{t \in I : f(t) = u, f'(t) > 0\}).$$

$\|f\|_p$ denotes the norm of f in $L^p(I, \lambda)$, $1 \leq p \leq +\infty$, λ the Lebesgue measure. The joint density of the finite set of random variables X_1, \dots, X_n at the point (x_1, \dots, x_n) will be denoted $p_{X_1, \dots, X_n}(x_1, \dots, x_n)$ whenever it exists. $\phi(t) := (2\pi)^{-1/2} \exp(-t^2/2)$ is the density of the standard normal distribution, $\Phi(t) := \int_{-\infty}^t \phi(u) du$ its distribution function. $|I|$ is the length of I . $x^+ = \sup\{x, 0\}$.

If A is a matrix, A^T denotes its transposed, and if A is a square matrix, $\det(A)$ its determinant. $Var(\xi)$ is the variance matrix of the (finite dimensional) random vector ξ and $Cov(\xi, \eta)$ the covariance of ξ and η .

For m and k , positive integers, $k \leq m$, define the factorial k th power of m by

$$m^{[k]} := m(m-1)\dots(m-k+1)$$

For other real values of m and k we put $m^{[k]} := 0$.

If k is an integer $k \geq 1$, the “diagonal of I^k ” is the set:

$$\mathcal{D}_k(I) := \{(t_1, \dots, t_k) \in I^k, t_j = t_h \text{ for some pair } (j, h), j \neq h\}.$$

$f^{(m)}$ is the m -th derivative of the function f . $\delta_{jh} = 0$ or 1 according as $j \neq h$ or $j = h$.

2 Rice formulae

In this section we review without proofs the formulae for the moments of the number of crossings of the paths of a stochastic process (the so-called “Rice formulae”) and some related results.

Let $X = \{X_t : t \in \mathbb{R}\}$ be a real valued stochastic process with C^1 paths. We set, for $(t_1, \dots, t_k) \in I^k \setminus \mathcal{D}_k(I)$ and $x_j \in \mathbb{R}$ ($j = 1, \dots, k$):

$$\tilde{A}_{t_1, \dots, t_k}(x_1, \dots, x_k) := \int_{\mathbb{R}^k} \left[\prod_{j=1}^k |x'_j| \right] p_{X_{t_1}, \dots, X_{t_k}, X'_{t_1}, \dots, X'_{t_k}}(x_1, \dots, x_k, x'_1, \dots, x'_k) dx'_1 \dots dx'_k$$

and

$$I_k(x_1, \dots, x_k) := \int_{I^k} \tilde{A}_{t_1, \dots, t_k}(x_1, \dots, x_k) dt_1 \dots dt_k,$$

where it is understood that the density in the integrand of the definition of $\tilde{A}_{t_1, \dots, t_k}(x_1, \dots, x_k)$ exists almost everywhere and that the integrals above can take the value $+\infty$.

Proposition 2.1 (Rice formula) *Let k be a positive integer, u a real number and I a bounded interval in the line. With the above notations and conditions, let us assume that the process X also satisfies the following conditions:*

1. *the density $p_{X_{t_1}, \dots, X_{t_k}, X'_{s_1}, \dots, X'_{s_k}}(x_1, \dots, x_k, x'_1, \dots, x'_k)$ exists for $(t_1, \dots, t_k), (s_1, \dots, s_k) \in I^k \setminus \mathcal{D}_k(I)$ and is a continuous function of (t_1, \dots, t_k) and of x_1, \dots, x_k at the point (u, \dots, u) .*

2. *the function $(t_1, \dots, t_k, x_1, \dots, x_k) \rightarrow \tilde{A}_{t_1, \dots, t_k}(x_1, \dots, x_k)$ is continuous for $(t_1, \dots, t_k) \in I^k \setminus \mathcal{D}_k(I)$ and x_1, \dots, x_k belonging to a neighbourhood of u .*

3. *(additional technical condition)*

$$\int_{\mathbb{R}^3} |x'_1|^{k-1} |x'_2 - x'_3| p_{X_{t_1}, \dots, X_{t_k}, X'_{s_1}, X'_{s_2}, X'_{s_1}}(x_1, \dots, x_k, x'_1, x'_2, x'_3) dx'_1 dx'_2 dx'_3 \rightarrow 0$$

as $|s_2 - t_1| \rightarrow 0$, uniformly as (t_1, \dots, t_k) varies in a compact subset of $I^k \setminus \mathcal{D}_k(I)$ and x_1, \dots, x_k vary in a fixed neighbourhood of u .

Then,

$$E((N_u(X, I))^{[k]}) = I_k(u, \dots, u) \quad (1)$$

Both members in (??) may be $+\infty$.

Remarks (a) For $k = 1$ the Rice formula (??) becomes

$$E[N_u(X; I)] = \int_I dt \int_{-\infty}^{+\infty} |x'| p_{X_t, X'_t}(u, x') dx'. \quad (2)$$

(b) Assume that the process X is Gaussian, centered and stationary with covariance function $\Gamma(t) := E(X_s X_{t+s})$, $s, t \in \mathbb{R}$ normalized by $\Gamma(0) = 1$, spectral measure

μ and spectral moments

$$\lambda_k := \int_{-\infty}^{+\infty} x^k \mu(dx) \quad (k = 0, 1, \dots)$$

whenever they exist. Then, (2) becomes

$$E[N_u(X; I)] = \frac{|I|}{\pi} \exp(-u^2/2) \lambda_2^{1/2}. \quad (3)$$

One can prove that formula (??) is in fact valid for any Gaussian, centered, stationary process having continuous paths in the sense that if $\lambda_2 < \infty$ equality holds and if $\lambda_2 = \infty$ the first member is $+\infty$ (Wschebor, 1985)

(c) A simple variation of (2), valid under the same hypotheses are:

$$E[U_u(X; I)] = \int_I dt \int_0^{+\infty} x' p_{X_t, X'_t}(u, x') dx' \quad (4)$$

In the same way one can obtain formulae for the factorial moments of “marked crossings”, that is, crossings such that some additional condition holds true. For example, if $Y = \{Y_t : t \in \mathbb{R}\}$ is some other stochastic process with real values such that for every t , the triplet (Y_t, X_t, X'_t) admits a density, $-\infty \leq a < b \leq +\infty$ and

$$N_u^{a,b}(X, I) := \#\{t : t \in I, X_t = u, a < Y_t < b\}.$$

Then

$$E[N_u^{a,b}(X; I)] = \int_a^b dy \int_I dt \int_{-\infty}^{+\infty} |x'| p_{Y_t, X_t, X'_t}(y, u, x') dx'. \quad (5)$$

Sufficient conditions for the validity of (2) are similar to those for Rice formula.

(d) The statement of Rice formula (??) in Proposition 2.1 is from Wschebor (1985). See Marcus (1977) for some extensions.

(e) It may be non trivial to verify the hypotheses of Proposition 2.1. However some general criteria are available. For example, if X is a Gaussian process with \mathcal{C}^1 paths and the densities

$$p_{X_{t_1}, \dots, X_{t_k}, X'_{s_1}, \dots, X'_{s_k}}$$

are non-degenerate for $(t_1, \dots, t_k) \in I^k \setminus \mathcal{D}_k(I)$, $(s_1, \dots, s_k) \in I^k \setminus \mathcal{D}_k(I)$, then conditions 1,2,3 of proposition 2.1 hold true (cf Wschebor (1985) p.37 for a proof, and also for some manageable sufficient conditions in non-Gaussian cases).

(f) Another question related to Formula (??) is the finiteness of the moments of crossings. There exists a series of results in the case of Gaussian processes (see for

example Belyaev (1966), Cramér and Leadbetter (1967), Cuzick (1975), Miroshin (1977)) in which the finiteness of $E((N_u(X, I))^k)$ is deduced from regularity conditions on the covariance. The proposition below contains sufficient conditions that can be reasonably checked in a general framework and which imply the finiteness of the moments of crossings.

Proposition 2.2 (Nualart and Wschebor, 1991) *Let $X = \{X_t : t \in I\}$ be a real valued stochastic process defined on the compact interval I of the real numbers, $u \in \mathbb{R}$ and m a positive integer. Suppose that a.s., the paths of X are of class \mathcal{C}^{p+1} , $p > 2m$ and that $p_{X_t}(x)$ is bounded by some constant C for x in a neighbourhood of u and all t in I . Denote $Z_h := \sup_{t \in I} |X_t^{(h)}|$. Then,*

$$E([N_u(X, I)]^m) \leq C_{p,m}[1 + E(Z_{p+1}) + C],$$

$C_{p,m}$ being some constant that depends only on p, m and the length of I .

The application of Proposition 2.2 to Gaussian processes is immediate. If X is Gaussian, with paths of class \mathcal{C}^{p+1} and $\text{Var}(X_t) > 0$, for all $t \in I$ then for $m < p/2$

$$E([N_u(X, I)]^m) < \infty \tag{6}$$

In fact, $E(Z_{p+1}) < \infty$ because of classical results (Fernique (1974)) so that (3) follows. As a consequence, if X is Gaussian with \mathcal{C}^∞ paths then (3) holds true for all $m = 1, 2, \dots$

3 The distribution of the maximum and the Rice series

We will assume in this section that the process X is defined on the interval $I = [0, 1]$, only for notational convenience. We introduce the notations

$$M := \max_{t \in [0,1]} X_t \quad ; \quad \nu_m := E((U_u)^{[m]} \mathbb{1}_{\{X_0 \leq u\}}) \quad (m = 1, 2, \dots)$$

where $U_u = U_u(X, I)$. ν_m is the factorial moment of the number of upcrossings of the process X with the level u on the interval $[0, 1]$, starting below u at $t = 0$. The

Rice formula to compute ν_m , whenever it holds is:

$$\begin{aligned} \nu_m &= \int_{I^m} dt_1 \dots dt_m \int_{-\infty}^u dx E \left((X'_{t_1})^+ \dots (X'_{t_m})^+ / X_0 = x, X_{t_1} = \dots = X_{t_m} = u \right) \\ & p_{X_0, X_{t_1}, \dots, X_{t_m}}(x, u, \dots, u) = \end{aligned} \quad (7)$$

$$\begin{aligned} &= \int_{I^m} dt_1 \dots dt_m \int_{-\infty}^u dx \int_{[0, +\infty)^m} x'_1 \dots x'_m \\ & p_{X_0, X_{t_1}, \dots, X_{t_m}, X'_{t_1}, \dots, X'_{t_m}}(x, u, \dots, u, x'_1, \dots, x'_m) dx'_1 \dots dx'_m. \end{aligned} \quad (8)$$

This section contains two main results. The first one, Theorem 3.1 requires the process to have C^∞ paths. As for Theorem 3.4, when the paths are not C^∞ , we first regularize them by means of a smoothing procedure and then apply a method that is similar to the one in the first theorem.

3.1 Smooth processes

Theorem 3.1 *Assume that a.s. the paths of the stochastic process X are of class C^∞ and that the density $p_{X_{1/2}}$ is bounded by some constant D .*

(i) *If there exists a sequence of positive numbers $\{c_k\}_{k=1,2,\dots}$ such that:*

$$\gamma_k := P(\|X^{(2k-1)}\|_\infty \geq c_k) + \frac{Dc_k}{2^{2k-1}(2k-1)!} = o(2^{-k}) \quad (k \rightarrow \infty) \quad (9)$$

then :

$$P(M > u) = P(X_0 > u) + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\nu_m}{m!} \quad (10)$$

(ii) *In formula (3.1) the error when one replaces the infinite sum by its m_0 -th partial sum is bounded by $\gamma_{m_0+1}^*$ where:*

$$\gamma_m^* := \sup_{k \geq m} (2^{k+1} \gamma_k).$$

We will call the series in the right-hand term of (3.1) the "Rice Series".

For the proof we start with the following lemma on the Cauchy remainder for polynomial interpolation (Davis 1975, Th 3.1.1).

Lemma 3.1 a) Let $I \subset \mathbb{R}$, let $f : I \rightarrow \mathbb{R}$ be of class \mathcal{C}^k , k a positive integer, t_1, \dots, t_k , k points in I and let $P(t)$ be the - unique - interpolation polynomial of degree $k - 1$ such that for $i = 1, \dots, k$: $f(t_i) = P(t_i)$, taking into account possible multiplicities.

Then, for $t \in I$:

$$f(t) - P(t) = \frac{1}{k!}(t - t_1)\dots(t - t_k)f^{(k)}(\xi)$$

where

$$\min(t_1, \dots, t_k, t) \leq \xi \leq \max(t_1, \dots, t_k, t).$$

b) If f is of class \mathcal{C}^k and has k zeros in I (taking into account possible multiplicities), then:

$$|f(1/2)| \leq \frac{1}{k!2^k} \|f^{(k)}\|_\infty.$$

The next combinatorial lemma plays a central role in what follows . Its statement is essentially known (Miroshin, 1974). We include an independent proof.

Lemma 3.2 Let ξ be a non-negative integer-valued random variable having finite moments of all orders. Let k, m, M ($k \geq 0, m \geq 1, M \geq 1$) be integers and denote

$$p_k := P(\xi = k) \quad ; \quad \mu_m := E(\xi^{[m]}) \quad ; \quad S_M := \sum_{m=1}^M (-1)^{m+1} \frac{\mu_m}{m!}$$

Then

(i) For each M :

$$S_{2M} \leq \sum_{k=1}^{2M} p_k \leq \sum_{k=1}^{\infty} p_k \leq S_{2M+1} \quad (11)$$

(ii) The sequence $\{S_M; M = 1, 2, \dots\}$ has a finite limit if and only if $\mu_m/m! \rightarrow 0$ as $m \rightarrow \infty$, and in that case:

$$P(\xi \geq 1) = \sum_{k=1}^{\infty} p_k = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\mu_m}{m!}. \quad (12)$$

Proof : (ii) is an immediate consequence of (i). As for (i) denote by $\binom{k}{m}$ the binomial numbers and write

$$S_M = \sum_{m=1}^M (-1)^{m+1} \sum_{k=m}^{\infty} \binom{k}{m} p_k = \sum_{k=1}^{\infty} p_k B_{k,M} \quad (13)$$

with

$$B_{k,M} := \sum_{m=1}^{k \wedge M} (-1)^{m+1} \binom{k}{m} \quad (14)$$

It is clear that $B_{k,M} = 1$ if $k \leq M$. If $k > M$, we have two cases:

1. $k \geq 2M$.

Note that $\binom{k}{m}$ increases with m if $1 \leq m \leq \frac{k}{2}$. It follows that

$B_{k,M} \geq k$ if M is odd and

$B_{k,M} \leq -\frac{k}{2}$ if M is even,

since $B_{k,M} \leq \binom{k}{1} - \binom{k}{2} \leq -k/2$, given that $k \geq 2M \geq 4$.

2. $M < k < 2M$.

Check that in this case

$$B_{k,M} = 1 + (-1)^{k+1} \sum_{h=0}^{k-M-1} (-1)^{h+1} \binom{k}{h} = 1 + (-1)^{k+1} (B_{k,k-M-1} - 1).$$

with the convention $B_{k,0} = 0$.

Since $k > 2(k-M-1)$, if $0 < k-M-1 < k$, we can apply the first case and it turns out that

$$k-M-1 \text{ odd} \Rightarrow B_{k,k-M-1} \geq k$$

$$k-M-1 \text{ even} \Rightarrow B_{k,k-M-1} \leq -k/2.$$

Finally if $k = M+1$, $B_{k,M} = 2$ when M is odd and $B_{k,M} = 0$ if M is even.

Summing up the two cases, if $k > M$, we have $B_{k,M} > 1$ if M is odd and $B_{k,M} \leq 0$ if M is even. So that from

$$S_M = \sum_{k=1}^M p_k + \sum_{k=M+1}^{\infty} p_k B_{k,M}$$

one gets (i). This proves the lemma.

Remark. A by-product of Lemma 3.2 that will be used in the sequel is the following: if in (??) one substitutes the infinite sum by the M -partial sum, the absolute value $\mu_{M+1}/((M+1)!)$ of the first neglected term is an upper-bound for the error in the computation of $P(\xi \geq 1)$.

Lemma 3.3 *With the same notations as in Lemma 3.2 we have the equality:*

$$E(\xi^{[m]}) = m \sum_{k=m}^{\infty} (k-1)^{[m-1]} P(\xi \geq k) \quad (m = 1, 2, \dots).$$

Proof: Check the identity

$$j^{[m]} = m \sum_{k=m-1}^{j-1} (k)^{[m-1]}$$

for each pair of integers j, m . So,

$$\begin{aligned} E(\xi^{[m]}) &= \sum_{j=m}^{\infty} j^{[m]} P(\xi = j) = \sum_{j=m}^{\infty} P(\xi = j) m \sum_{k=m}^j (k-1)^{[m-1]} = \\ &= m \sum_{k=m}^{\infty} (k-1)^{[m-1]} P(\xi \geq k). \end{aligned}$$

Lemma 3.4 *Suppose that a.s. the paths of the process X belong to \mathcal{C}^{∞} and that $p_{X_{1/2}}$ is bounded by the constant D . Then for any sequence $\{c_k, k = 1, 2, \dots\}$ of positive numbers, one has*

$$E((U_u)^{[m]}) \leq m \sum_{k=m}^{\infty} (k-1)^{[m-1]} \left[P(\|X^{(2k-1)}\|_{\infty} \geq c_k) + \frac{Dc_k}{2^{2k-1} (2k-1)!} \right], \quad (15)$$

Proof: Because of Lemma 3.3 it is enough to prove that $P(U_u \geq k)$ is bounded by the expression in brackets in the right-hand member of (3.1). We have

$$P(U_u \geq k) \leq P(\|X^{(2k-1)}\|_{\infty} \geq c_k) + P(U_u \geq k, \|X^{(2k-1)}\|_{\infty} < c_k).$$

Because of Rolle's theorem:

$$\{U_u \geq k\} \subset \{N_u(X; I) \geq 2k-1\},$$

Applying Lemma 3.1 to the function $X_{(\cdot)} - u$ and replacing in its statement k by $2k - 1$, we obtain:

$$\{U_u \geq k, \|X^{(2k-1)}\|_\infty < c_k\} \subset \{|X_{1/2} - u| \leq \frac{c_k}{2^{2k-1} (2k-1)!}\}.$$

The remaining is plain.

Proof of Theorem 3.1 :

Introduce the notation $\tilde{\nu}_m := E(U_u^{[m]})$ ($m = 1, 2, \dots$). Using Lemma 3.4 and the hypothesis we obtain:

$$\frac{\tilde{\nu}_m}{m!} \leq \frac{1}{m!} \sum_{k=m}^{\infty} k^{[m]} \gamma_m^* 2^{-(k+1)} = \frac{\gamma_m^*}{m!} 2^{-(m+1)} \left[\left(\frac{1}{1-x} \right)^{(m)} \Big|_{x=1/2} \right] = \gamma_m^*$$

Since $\nu_m \leq \tilde{\nu}_m$ we can apply Lemma 3.2 to the random variable $\xi = U_u \mathbb{1}_{\{X_0 \leq u\}}$ and the result follows from $\gamma_m^* \rightarrow 0$.

Remarks

One can replace the condition $p_{X_{1/2}}(x) \leq D$ for all x by $p_{X_{1/2}}(x) \leq D$ for x in some neighbourhood of u . In this case, the statement of Theorem 3.1 holds if one adds in (ii) that the error is bounded by $\gamma_{m_0+1}^*$ for m_0 large enough. The proof is similar. Also, one may substitute the one-dimensional density $p_{X_{1/2}}$ by p_{X_t} for some other $t \in I$, introducing into the bounds the corresponding modifications.

The application of Theorem 3.1 requires an adequate choice of the sequence $\{c_k, k = 1, 2, \dots\}$ that depends on the available description of the process X . The whole procedure will have some practical interest for the computation of $P(M > u)$ only if we get appropriate bounds for the quantities γ_m^* and the factorial moments ν_m can be actually computed by means of Rice formulae (or by some other procedure). Below we show how this can be done for some classes of processes.

Theorem 3.2 *Let X be Gaussian, centered and stationary. Suppose that the spectral moments λ_k ($k = 0, 1, \dots$) are finite, $\lambda_0 = 1$ and moreover*

$$H_1) \quad \lambda_{2k} \leq C_1 (k!)^2,$$

for some constant C_1 . Then the process satisfies the conclusion of Theorem 3.1 so that the Rice series converges and gives the distribution function of M .

Proof. It is clear that $p_{X_{1/2}}(x) \leq D = (2\pi)^{-1/2}$. Let us show a sequence $\{c_k\}$ that satisfies (??). We have

$$\begin{aligned} P(\|X^{(2k-1)}\|_\infty \geq c_k) &\leq P(|X_0^{(2k-1)}| \geq c_k) + 2P(U_{c_k}(X^{(2k-1)}, I) \geq 1) \leq \\ &\leq P(|Z| \geq c_k \lambda_{4k-2}^{-1/2}) + 2E(U_{c_k}(X^{(2k-1)}, I)), \end{aligned} \quad (16)$$

where Z is standard normal. Note that $\{X_t^{(2k-1)}; t \in \mathbb{R}\}$ is a Gaussian stationary centered process with covariance function $-\Gamma^{(4k-2)}(\cdot)$. So we can use Rice formula (??) to compute the second term in the right-hand member of (16), taking into account that one has to rescale the process by a factor $\lambda_{4k-2}^{1/2}$ and that we have upcrossings instead of crossings. Using the inequality $1 - \Phi(x) \leq (1/x)\phi(x)$ valid for $x > 0$, one gets:

$$P(\|X^{(2k-1)}\|_\infty \geq c_k) \leq \left[\sqrt{\frac{2}{\pi}} \frac{\lambda_{4k-2}^{1/2}}{c_k} + (1/\pi) \left(\frac{\lambda_{4k}}{\lambda_{4k-2}} \right)^{1/2} \right] \exp\left(-\frac{c_k^2}{2\lambda_{4k-2}}\right)$$

Choose

$$\begin{aligned} c_k &:= (B_1 k \lambda_{4k-2})^{1/2} \text{ if } \frac{\lambda_{4k}}{\lambda_{4k-2}} \leq B_1 k \\ c_k &:= (\lambda_{4k})^{1/2} \text{ if } \frac{\lambda_{4k}}{\lambda_{4k-2}} > B_1 k. \end{aligned}$$

Using hypothesis H_1), if $B_1 > 1$:

$$P(\|X^{(2k-1)}\|_\infty \geq c_k) \leq \left[\sqrt{\frac{2}{\pi}} + \frac{1}{\pi} (B_1 k)^{1/2} \right] e^{-\frac{B_1 k}{2}}.$$

Finally, choosing $B_1 := 4 \log(2)$:

$$\gamma_k \leq \sqrt{\frac{2}{\pi}} (1 + 2(C_1^{1/2} + 1)k) 2^{-2k} \quad (k = 1, 2, \dots),$$

so that (??) is satisfied. As a by product, note that

$$\gamma_m^* \leq \sqrt{\frac{8}{\pi}} (1 + 2(C_1^{1/2} + 1)m) 2^{-m} \quad (m = 1, 2, \dots). \quad (17)$$

Remarks

a) One can easily verify that condition H_1) implies that the covariance Γ can be extended to an analytic function in the disc $\{z : |z| < 2\}$ of the complex plane and that - almost conversely - if the Taylor series of Γ at the origin is absolutely convergent at $z = 2$ then H_1) holds true.

b) On the other hand, if one is willing to use Rice formulae to compute the factorial moments ν_m , from remark e) after Proposition 2.1, it is enough to verify that the distribution of

$$X_{t_1}, \dots, X_{t_k}, X'_{t_1}, \dots, X'_{t_k}$$

is non-degenerate for any choice of $k = 1, 2, \dots$ $(t_1, \dots, t_k) \in I^k \setminus \mathcal{D}_k(I)$. For Gaussian stationary processes a sufficient condition for non-degeneracy is the spectral measure not to be purely atomic. See Cramér and Leadbetter (1967) for a proof. The same kind of argument permits to show that the conclusion remains if the spectral measure is purely atomic and the set of its atoms has an accumulation point in \mathbb{R} .

If instead of requiring the paths of the process X to be of class \mathcal{C}^∞ , one relaxes this condition up to a certain order of differentiability, one can still get upper and lower bounds for $P(M > u)$. The proof of the following statement is a straightforward application of Lemma 3.2 and Proposition 2.2.

Theorem 3.3 *Let $X = \{X_t : t \in I\}$ be a real -valued stochastic process. Suppose that $p_{X_t}(x)$ is bounded for $t \in I$, $x \in \mathbb{R}$ and that the paths of X are of class \mathcal{C}^{p+1} . Then*

$$\text{if } 2K + 1 < p/2 : P(M > u) \leq P(X_0 > u) + \sum_{m=1}^{2K+1} (-1)^{m+1} \frac{\nu_m}{m!}$$

and

$$\text{if } 2K < p/2 : P(M > u) \leq P(X_0 > u) + \sum_{m=1}^{2K} (-1)^{m+1} \frac{\nu_m}{m!}.$$

Note that all the moments in the above formulae are finite.

When the level u is high, the results by Piterbag (1981, 1996), which were until recently the sharpest known asymptotic bounds for the tail of the distribution of the maximum on a fixed interval of general Gaussian stationary processes with regular paths (see Bardet and Wschebor, 2000) for a refinement) can be deduced from the foregoing arguments. Here, only the *first* term in the Rice series takes part in the equivalent of $P(M > u)$ as $u \rightarrow +\infty$. More precisely, if $\lambda_4 < \infty$, it is not hard to prove that

$$0 \leq \sqrt{\frac{\lambda_2}{2\pi}}\phi(u) - \nu_1 \leq (\text{const})e^{-\frac{u^2(1+\eta)}{2}},$$

$$\nu_2 \leq (\text{const})e^{-\frac{u^2(1+\eta)}{2}},$$

for a certain $\eta > 0$. Lemma 3.2 implies that

$$0 \leq 1 - \Phi(u) + \sqrt{\frac{\lambda_2}{2\pi}}\phi(u) - P(M > u) \leq (\text{const})e^{-\frac{u^2(1+\eta)}{2}}, \quad (18)$$

which is Piterbarg's result.

3.2 Computational aspects for smooth stationary Gaussian processes

Next, our aim is to compare the numerical computation based upon Theorem 3.1 with the Monte-Carlo method based on the simulation of the paths. We do this for stationary Gaussian processes that satisfy the hypotheses of Theorem 3.2 and also the non-degeneracy condition that ensures that one is able to compute the factorial moments of crossings by means of Rice formulae (see remark b) after the statement of this Theorem). We also include some examples in which the distribution of the maximum is computed by means of Theorem 3.3.

Suppose that we want to compute $P(M > u)$ with an error bounded by δ , where $\delta > 0$ is a given positive number.

To proceed by simulation, we discretize the paths by means of a uniform partition $\{t_j := j/n, j = 0, 1, \dots, n\}$. Denote

$$M^{(n)} := \sup_{0 \leq j \leq n} X_{t_j}.$$

Using Taylor's formula at the time where the maximum M of $X_{(\cdot)}$ occurs, one gets :

$$0 \leq M - M^{(n)} \leq \|X''\|_{\infty}/(2n^2)$$

It follows that

$$0 \leq P(M > u) - P(M^{(n)} > u) = P(M > u, M^{(n)} \leq u) \leq \\ \leq P(u < M \leq u + \|X''\|_{\infty}/(2n^2)).$$

If we admit that the distribution of M has a locally bounded density (which is a well-known fact under the mentioned hypothesis) the above suggests that a number of $n = (\text{const})\delta^{-1/2}$ points is required if one wants the mean error $P(M > u) - P(M^{(n)} > u)$ to be bounded by δ .

On the other hand, to estimate $P(M^{(n)} > u)$ by Monte-Carlo with a *mean square error* smaller than δ , we require the simulation of $N = (\text{const})\delta^{-2}$ Gaussian n-tuples $(X_{t_1}, \dots, X_{t_n})$ from the distribution determined by the given stationary process. Performing each simulation demands $(\text{const})n \log(n)$ elementary operations (Dietrich and Newsam, 1997). Summing up, the total mean number of elementary operations required to get a *mean square error* bounded by δ in the estimation of $P(M > u)$ has the form $(\text{const})\delta^{-5/2} \log(1/\delta)$.

Suppose now that we apply Theorem 3.1 to a Gaussian stationary centered process verifying hypotheses H_1) and the non-degeneracy condition. The bound for γ_m^* in Equation (3.1) implies that computing a partial sum with $(\text{const})\log(1/\delta)$ terms assures that the tail in the Rice series is bounded by δ . If one computes each ν_m by means of a Monte-Carlo method for the multiple integrals appearing in the Rice formulae, then the number of elementary operations for the whole procedure will have the form $(\text{const})\delta^{-2} \log(1/\delta)$. Hence, this is better than simulation as δ tends to zero.

As usual, for given $\delta > 0$, the value of the generic constants decides the comparison between both methods.

More important is the fact that the enveloping property of the Rice series implies that the actual number of terms required by the application of Theorem 3.1 can be much smaller than the one resulting from the a priori bound on γ_m^* . More precisely, suppose that we have obtained each numerical approximation ν_m^* of ν_m with a precision η

$$|\nu_m^* - \nu_m| \leq \eta,$$

and that we stop when

$$\frac{\nu_{m_0+1}^*}{(m_0 + 1)!} \leq \eta. \tag{19}$$

Then, it follows that

$$\left| \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\nu_m}{m!} - \sum_{m=1}^{m_0} (-1)^{m+1} \frac{\nu_m^*}{m!} \right| \leq (e + 1)\eta.$$

Putting $\eta = \delta/(e + 1)$, we get the desired bound. In other words one can profit of the successive numerical approximations of ν_m to determine a new m_0 which turns

out to be - in certain interesting examples - much smaller than the one deduced from the a priori bound on γ_m^* .

3.2.1 Computation of moments.

An efficient numerical computation of the factorial moments of crossings is associated to a fine description of the behaviour as the k -tuple (t_1, \dots, t_k) approaches the diagonal $\mathcal{D}_k(I)$, of the integrands

$$\tilde{A}_{t_1, \dots, t_k}^+(u, \dots, u) = \int_{[0, +\infty)^m} x'_1 \dots x'_m p_{X_{t_1}, \dots, X_{t_m}, X'_{t_1}, \dots, X'_{t_m}}(u, \dots, u, x'_1, \dots, x'_m) dx'_1 \dots dx'_m.$$

$$A_{t_1, \dots, t_k}(u) = \int_{-\infty}^u dx \int_{[0, +\infty)^m} x'_1 \dots x'_m p_{X_0, X_{t_1}, \dots, X_{t_m}, X'_{t_1}, \dots, X'_{t_m}}(x, u, \dots, u, x'_1, \dots, x'_m) dx'_1 \dots dx'_m.$$

that appear respectively in Rice formulae for the k -factorial moment of upcrossings and the k -factorial moment of upcrossings with the additional condition that $X_0 \leq u$.

For example in Azaïs, Cierco and Croquette (1999) it is proved that if X is Gaussian, stationary, centered and λ_8 is finite, then the integrand $\tilde{A}_{s,t}^+(u, u)$ in the computation of $\tilde{\nu}_2$ - the second factorial moment of the number of upcrossings - satisfies:

$$\tilde{A}_{s,t}^+(u, u) \simeq \frac{1}{1296} \frac{(\lambda_2 \lambda_6 - \lambda_4^2)^{3/2}}{(\lambda_4 - \lambda_2^2)^{1/2} \pi^2 \lambda_2^2} \exp\left(-\frac{1}{2} \frac{\lambda_4}{\lambda_4 - \lambda_2^2} u^2\right) (t - s)^4, \quad (20)$$

as $t - s \rightarrow 0$.

(??) can be extended to non-stationary Gaussian processes obtaining an equivalence of the form:

$$\tilde{A}_{s,t}^+(u, u) \simeq J(\tilde{t})(t - s)^4 \quad \text{as } s, t \rightarrow \tilde{t} \quad (21)$$

where $J(\tilde{t})$ is a continuous non-zero function of \tilde{t} depending on u , that can be expressed in terms of the mean and covariance functions of the process and its derivatives. We give a proof of an equivalence of the form (??) in the next proposition.

One can profit of this equivalence to improve the numerical methods to compute ν_2 (the second factorial moment of the number of upcrossings restricted to $X_0 \leq u$). Equivalence formulae such as (??) or (??) can be used to avoid numerical degeneracies near the diagonal $D_2(I)$. Note that even in case X is stationary at the departure, under conditioning on X_0 , the process that must be taken into account in

the actual computation of the factorial moments of upcrossings that appear in the Rice series(3.1) will be non-stationary, so that equivalence (??) is the appropriate tool.

Proposition 3.1 *Suppose that X is a Gaussian process with C^5 paths and that for each $t \in I$ the joint distribution of $X_t, X'_t, X_t^{(2)}, X_t^{(3)}$ does not degenerate. Then (??) holds true.*

Proof. Denote by $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ a two-dimensional random vector having as probability distribution the conditional distribution of $\begin{pmatrix} X'_s \\ X'_t \end{pmatrix}$ given $X_s = X_t = u$.

One has:

$$\tilde{A}_{s,t}^+(u, u) = E(\xi_1^+ \xi_2^+) p_{X_s, X_t}(u, u) \quad (22)$$

Put $\tau = t - s$ and check the following Taylor expansions around the point s :

$$E(\xi_1) = m_1\tau + m_2\tau^2 + L_1\tau^3 \quad (23)$$

$$E(\xi_2) = -m_1\tau + m'_2\tau^2 + L_2\tau^3 \quad (24)$$

$$\text{Var}(\xi) = \begin{pmatrix} a\tau^2 + b\tau^3 + c\tau^4 + \rho_{11}\tau^5 & -a\tau^2 - \frac{b+b'}{2}\tau^3 + d\tau^4 + \rho_{12}\tau^5 \\ -a\tau^2 - \frac{b+b'}{2}\tau^3 + d\tau^4 + \rho_{12}\tau^5 & a\tau^2 + b'\tau^3 + c'\tau^4 + \rho_{22}\tau^5 \end{pmatrix} \quad (25)$$

where $m_1, m_2, m'_2, a, b, c, d, a', b', c'$ are continuous functions of s and $L_1, L_2, \rho_{11}, \rho_{12}, \rho_{22}$ are bounded functions of s and t . (??),(??) and (3.2.1) follow directly from the regression formulae of the pair $\begin{pmatrix} X'_s \\ X'_t \end{pmatrix}$ on the condition $X_s = X_t = u$.

Note that (as in Belyaiev, 1966 or Azaïs and Wschebor, 2000)

$$\text{Var}(\xi_1) = \frac{\det \text{Var}(X_s, X_t, X'_s)^T}{\det \text{Var}(X_s, X_t)^T} = \frac{\det \text{Var}(X_s, X'_s, X_t - X_s - (t-s)X'_s)^T}{\det \text{Var}(X_s, X_t - X_s)^T}$$

A direct computation gives:

$$\det \text{Var}(X_s, X_t)^T \approx \tau^2 \det \text{Var}(X_s, X'_s)^T \quad (26)$$

$$\text{Var}(\xi_1) \approx \frac{1}{4} \frac{\det \text{Var}(X_s, X'_s, X_s^{(2)})^T}{\det \text{Var}(X_s, X'_s)^T} \tau^2$$

where \approx denotes equivalence as $\tau \rightarrow 0$. So,

$$a = \frac{1}{4} \frac{\det \text{Var}(X_s, X'_s, X_s^{(2)})^T}{\det \text{Var}(X_s, X'_s)^T}$$

which is a continuous non-vanishing function for $s \in I$. Note that the coefficient of τ^3 in the Taylor expansion of $Cov(\xi_1, \xi_2)$ is equal to $-\frac{b+b'}{2}$. This follows either by direct computation or noting that $\det Var(\xi)$ is a symmetric function of the pair s, t .

Put

$$\Delta(s, t) = \det Var(\xi)$$

The behaviour of $\Delta(s, t)$ as $s, t \rightarrow \tilde{t}$ can be obtained by noting that

$$\Delta(s, t) = \frac{\det Var(X_s, X_t, X'_s, X'_t)^T}{\det Var(X_s, X_t)^T}$$

and applying Lemma 3.2 in Azaïs and Wschebor (2000) or Lemma 4.3, p.76 in Piterbarg (1996) which provide an equivalent for the numerator, so that:

$$\Delta(s, t) \approx \bar{\Delta}(\tilde{t})\tau^6 \quad (27)$$

with

$$\bar{\Delta}(\tilde{t}) = \frac{1}{144} \frac{\det Var(X_{\tilde{t}}, X'_{\tilde{t}}, X_{\tilde{t}}^{(2)}, X_{\tilde{t}}^{(3)})^T}{\det Var(X_{\tilde{t}}, X'_{\tilde{t}})^T}$$

The hypothesis implies that $\bar{\Delta}(\tilde{t})$ is continuous and non zero.

Then:

$$E(\xi_1^+ \xi_2^+) = \frac{1}{2\pi [\Delta(s, t)]^{1/2}} \int_0^{+\infty} \int_0^{+\infty} xy \exp \left[-\frac{1}{2\Delta(s, t)} F(x, y) \right] dx dy \quad (28)$$

where

$$F(x, y) = Var(\xi_2)(x - E(\xi_1))^2 + Var(\xi_1)(y - E(\xi_2))^2 - 2Cov(\xi_1, \xi_2)(x - E(\xi_1))(y - E(\xi_2))$$

Substituting the expansions (??), (??), (3.2.1) in the integrand of (3.2.1) and making the change of variables $x = \tau^2 v$, $y = \tau^2 w$ we get, as $s, t \rightarrow \tilde{t}$:

$$E(\xi_1^+ \xi_2^+) \approx \frac{\tau^5}{2\pi [\bar{\Delta}(\tilde{t})]^{1/2}} \int_0^{+\infty} \int_0^{+\infty} vw \exp \left[-\frac{1}{2\bar{\Delta}(\tilde{t})} \bar{F}(v, w) \right] dv dw \quad (29)$$

$\bar{\Delta}(\tilde{t})$ can be also expressed in terms of the functions a, b, c, d, a', b', c' :

$$\bar{\Delta}(\tilde{t}) = ac' + ca' + 2ad - \left(\frac{b - b'}{2} \right)^2$$

and

$$\overline{F}(v, w) = a(v - m_2 + w - m'_2)^2 + m_1^2(c + c' + 2d) - m_1(b - b')(v + w - m_2 - m'_2)$$

The functions $a, b, c, d, b', c', m_1, m_2$ that appear in these formulae are all evaluated at the point \tilde{t} .

Replacing (3.2.1) and (3.2.1) into (??) one gets (??).

For $k \geq 3$, the general behaviour of the functions $A_{t_1, \dots, t_k}(u)$ and $\tilde{A}_{t_1, \dots, t_k}^+(u, \dots, u)$ when (t_1, \dots, t_k) approaches the diagonal is not known. Proposition 3.3, even though it contains restrictive conditions (it requires $E\{X_t\} = 0$ and $u = 0$) can be applied to improve the efficiency in the computation of the k -th factorial moments by means of a Monte-Carlo method, via the use of important sampling. More precisely, when computing the integral of $\tilde{A}_{t_1, \dots, t_k}^+(u)$ over I^k , instead of choosing at random the point (t_1, t_2, \dots, t_k) in the cube I^k with a uniform distribution, we do it with a probability law that has a density proportional to the function $\prod_{1 \leq i < j \leq k} (t_j - t_i)^4$. For its proof we will use the following auxiliary proposition, that has its own interest and extends (3.2.1) to any k .

Proposition 3.2 *Suppose that $X = \{X_t : t \in I\}$ is a Gaussian process defined on the interval I of the real line with C^{2k-1} paths, k an integer, $k \geq 2$, and that the joint distribution of $(X_t, X'_t, \dots, X_t^{(2k-1)})$ is non-degenerate for each $t \in I$. Then, if $t_1, t_2, \dots, t_k \rightarrow t^*$:*

$$\Delta = \det \text{Var}(X_{t_1}, X'_{t_1}, \dots, X_{t_k}, X'_{t_k})^T \approx \frac{\det \text{Var}(X_{t^*}, X'_{t^*}, \dots, X_{t^*}^{(2k-1)})^T}{[2! \cdot 3! \cdot \dots \cdot (2k-1)!]^2} \prod_{1 \leq i < j \leq k} (t_j - t_i)^8 \quad (30)$$

Proof. With no loss of generality, we consider only k -tuples (t_1, t_2, \dots, t_k) such that $t_i \neq t_j$ if $i \neq j$.

Suppose $f : I \rightarrow \mathbb{R}$ is a function of class C^{2m} , $m \geq 1$, and t_1, t_2, \dots, t_m are pairwise different points in I . We use the following notations for interpolating polynomials:

$P_m(t; f)$ is the polynomial of degree $2m - 1$ such that

$$P_m(t_j; f) = f(t_j) \quad \text{and} \quad P'_m(t; f) = f'(t_j) \quad \text{for } j = 1, \dots, m.$$

$Q_m(t; f)$ is the polynomial of degree $2m - 2$ such that

$$Q_m(t_j; f) = f(t_j) \quad \text{for } j = 1, \dots, m; \quad Q'_m(t; f) = f'(t_j) \quad \text{for } j = 1, \dots, m - 1.$$

From Lemma 3.1 we know that

$$f(t) - P_m(t; f) = \frac{1}{(2m)!} (t - t_1)^2 \dots (t - t_m)^2 f^{(2m)}(\xi) \quad (31)$$

$$f(t) - Q_m(t; f) = \frac{1}{(2m-1)!} (t - t_1)^2 \dots (t - t_{m-1})^2 (t - t_m) f^{(2m-1)}(\eta) \quad (32)$$

where

$$\xi = \xi(t_1, t_2, \dots, t_m, t), \eta = \eta(t_1, t_2, \dots, t_m, t)$$

and

$$\min(t_1, t_2, \dots, t_m, t) \leq \xi, \eta \leq \max(t_1, t_2, \dots, t_m, t).$$

Note that the function

$$g(t) = f^{(2m-1)}(\eta(t_1, t_2, \dots, t_m, t)) = \frac{(2m-1)! [f(t) - Q_m(t; f)]}{(t - t_1)^2 \dots (t - t_{m-1})^2 (t - t_m)}$$

is differentiable at the point $t = t_m$ and differentiating in (3.2.1):

$$f'(t_m) - Q'_m(t_m; f) = \frac{1}{(2m-1)!} (t_m - t_1)^2 \dots (t_m - t_{m-1})^2 f^{(2m-1)}(\eta(t_1, t_2, \dots, t_m, t_m)) \quad (33)$$

Put

$$\xi_m = \xi(t_1, t_2, \dots, t_m, t_m), \eta_m = \eta(t_1, t_2, \dots, t_m, t_m).$$

Since $P_m(t; f)$ is a linear functional of

$$(f(t_1), \dots, f(t_m), f'(t_1), \dots, f'(t_m))$$

and $Q_m(t; f)$ is a linear functional of

$$(f(t_1), \dots, f(t_m), f'(t_1), \dots, f'(t_{m-1}))$$

with coefficients depending (in both cases) only on t_1, t_2, \dots, t_m, t , it follows that:

$$\begin{aligned} \Delta &= \det \text{Var}(X_{t_1}, X'_{t_1}, X_{t_2} - P_1(t_2; X), X'_{t_2} - Q'_1(t_2, X), \dots \\ &\quad \dots, X_{t_k} - P_{k-1}(t_k; X), X'_{t_k} - Q'_k(t_k; X))^T = \\ &= \det \text{Var}(X_{t_1}, X'_{t_1}, \frac{1}{2!}(t_2 - t_1)^2 X_{\xi_1}^{(2)}, \frac{1}{3!}(t_2 - t_1)^2 X_{\eta_2}^{(3)}, \dots \\ \dots, \frac{1}{(2k-2)!}(t_k - t_1)^2 \dots (t_k - t_{k-1})^2 X_{\xi_{k-1}}^{(2k-2)}, \frac{1}{(2k-1)!}(t_k - t_1)^2 \dots (t_k - t_{k-1})^2 X_{\eta_{k-1}}^{(2k-1)})^T = \\ &= \frac{\tilde{\Delta}}{[2! \dots (2k-1)!]^2} \prod_{1 \leq i < j \leq k} (t_j - t_i)^8 \end{aligned}$$

with

$$\begin{aligned}\tilde{\Delta} &= \det \text{Var}(X_{t_1}, X'_{t_1}, X_{\xi_1}^{(2)}, X_{\eta_2}^{(3)}, \dots, X_{\xi_{k-1}}^{(2k-2)}, X_{\eta_{k-1}}^{(2k-1)})^T \rightarrow \\ &\rightarrow \det \text{Var}(X_{t^*}, X'_{t^*}, \dots, X_{t^*}^{(2k-1)})^T\end{aligned}$$

as $t_1, t_2, \dots, t_k \rightarrow t^*$. This proves (3.2.1).

Proposition 3.3 *Suppose that X is a centered Gaussian process with C^{2k-1} paths and that for each pairwise distinct values of the parameter $t_1, t_2, \dots, t_k \in I$ the joint distribution of $(X_{t_h}, X'_{t_h}, \dots, X_{t_h}^{(2k-1)})$, $h = 1, 2, \dots, k$ is non-degenerate. Then, as $t_1, t_2, \dots, t_k \rightarrow t^*$:*

$$\tilde{A}_{t_1, \dots, t_k}^+(0, \dots, 0) \approx J_k(t^*) \prod_{1 \leq i < j \leq k} (t_j - t_i)^4$$

where $J_k(t)$ is a continuous non-zero function of t .

Proof. Introduce the notation

$$D_k(t) = \det \text{Var}(X_t, X'_t, \dots, X_t^{(k)})^T$$

In the same way as in the proof of Proposition 3.2 and with a simpler computation, it follows that as $t_1, t_2, \dots, t_k \rightarrow t^*$

$$\det \text{Var}(X_{t_1}, X_{t_2}, \dots, X_{t_k})^T \approx \frac{1}{[2! \dots (k-1)!]^2} \left[\prod_{1 \leq i < j \leq k} (t_j - t_i)^2 \right] \cdot D_{k-1}(t^*). \quad (34)$$

For pairwise different values t_1, t_2, \dots, t_k , let $Z = (Z_1, \dots, Z_k)^T$ be a random vector having the conditional distribution of $(X'_{t_1}, \dots, X'_{t_k})^T$ given $X_{t_1} = X_{t_2} = \dots = X_{t_k} = 0$. The (Gaussian) distribution of Z is centered and we denote its covariance matrix by Σ . Also put:

$$\Sigma^{-1} = \frac{1}{\det(\Sigma)} (\sigma^{ij})_{i,j=1, \dots, k}$$

σ^{ij} being the cofactor of the position (i, j) in the matrix Σ . Then, one can write:

$$\tilde{A}_{t_1, \dots, t_k}^+(0, \dots, 0) = E \{Z_1^+ \dots Z_k^+\} \cdot p_{X_{t_1}, \dots, X_{t_k}}(0, \dots, 0) \quad (35)$$

and

$$\tilde{A}_{t_1, \dots, t_k}^+(0, \dots, 0) = \frac{1}{(2\pi)^{\frac{k}{2}} (\det(\Sigma))^{\frac{1}{2}}} \int_{(R^+)^k} x_1 \dots x_k \exp \left[-\frac{F(x_1, \dots, x_k)}{2 \cdot \det(\Sigma)} \right] dx_1 \dots dx_k \quad (36)$$

where

$$F(x_1, \dots, x_k) = \sum_{i,j=1}^k \sigma^{ij} x_i x_j.$$

Letting $t_1, t_2, \dots, t_k \rightarrow t^*$ and using (3.2.1) and (3.2.1) we get:

$$\begin{aligned} \det(\Sigma) &= \frac{\det \text{Var}(X_{t_1}, X'_{t_1}, \dots, X_{t_k}, X'_{t_k})^T}{\det \text{Var}(X_{t_1}, \dots, X_{t_k})^T} \approx \\ &\approx \frac{1}{[k! \dots (2k-1)!]^2} \left[\prod_{1 \leq i < j \leq k} (t_j - t_i)^6 \right] \cdot \frac{D_{2k-1}(t^*)}{D_{k-1}(t^*)}. \end{aligned}$$

We consider now the behaviour of the σ^{ij} ($i, j = 1, \dots, k$). Let us first look at σ^{11} . Using the same method as above, now applied to the cofactor of the position $(1, 1)$ in Σ , one has:

$$\begin{aligned} \sigma^{11} &= \frac{\det \text{Var}(X_{t_1}, X_{t_2}, \dots, X_{t_k}, X'_{t_2}, \dots, X'_{t_k})^T}{\det \text{Var}(X_{t_1}, \dots, X_{t_k})^T} \approx \\ &\approx \frac{\frac{1}{[2! \dots (2k-2)!]^2} \left[\prod_{2 \leq i < j \leq k} (t_j - t_i)^8 \right] \left[\prod_{2 \leq h \leq k} (t_1 - t_h)^4 \right] D_{2k-2}(t^*)}{\frac{1}{[2! \dots (k-1)!]^2} \left[\prod_{1 \leq i < j \leq k} (t_j - t_i)^2 \right] D_{k-1}(t^*)} = \\ &= \frac{1}{[k! \dots (2k-2)!]^2} \left[\prod_{2 \leq i < j \leq k} (t_j - t_i)^6 \right] \left[\prod_{2 \leq h \leq k} (t_1 - t_h)^2 \right] \frac{D_{2k-2}(t^*)}{D_{k-1}(t^*)} \end{aligned}$$

A similar computation holds for σ^{ii} , $i = 2, \dots, k$.

Consider now σ^{12} . One has:

$$\begin{aligned} \sigma^{12} &= - \frac{\det [E \{ (X_{t_1}, X_{t_2}, \dots, X_{t_k}, X'_{t_2}, \dots, X'_{t_k})^T \cdot (X_{t_1}, X_{t_2}, \dots, X_{t_k}, X'_{t_1}, X'_{t_3}, \dots, X'_{t_k}) \}]}{\det \text{Var}(X_{t_1}, \dots, X_{t_k})^T} = \\ &= \frac{\det [E \{ (X_{t_2}, X'_{t_2}, \dots, X_{t_k}, X'_{t_k}, X_{t_1})^T \cdot (X_{t_1}, X'_{t_1}, X_{t_3}, X'_{t_3}, \dots, X_{t_k}, X'_{t_k}, X_{t_2}) \}]}{\det \text{Var}(X_{t_1}, \dots, X_{t_k})^T} \approx \\ &\approx \frac{1}{[k! \dots (2k-2)!]^2} \left[\prod_{3 \leq i < j \leq k} (t_j - t_i)^6 \right] \left[\prod_{3 \leq h \leq k} (t_1 - t_h)^4 (t_2 - t_h)^4 \right] (t_2 - t_1)^2 \cdot \frac{D_{2k-2}(t^*)}{D_{k-1}(t^*)} \end{aligned}$$

A similar computation applies to all the cofactors σ^{ij} , $i \neq j$.

Now we perform in the integral in (3.2.1) the change of variables

$$x_j = \left[\prod_{i=1, i \neq j}^{i=k} (t_i - t_j)^2 \right] \cdot y_j \quad j = 1, \dots, k$$

and the integral becomes:

$$\left[\prod_{1 \leq i < j \leq k} (t_j - t_i)^8 \right] \int_{(R^+)^k} y_1 \dots y_k \exp \left[-\frac{1}{2 \cdot \det(\Sigma)} G(y_1, \dots, y_k) \right] dy_1 \dots dy_k$$

where

$$G(y_1, \dots, y_k) = \sum_{i,j=1}^k \sigma^{ij} \left[\prod_{h=1, h \neq i}^{h=k} (t_h - t_i)^2 \right] \left[\prod_{h=1, h \neq j}^{h=k} (t_h - t_j)^2 \right] y_i y_j.$$

so that, as $t_1, t_2, \dots, t_k \rightarrow t^*$

$$\frac{G(y_1, \dots, y_k)}{\det(\Sigma)} \approx [(2k-1)!]^2 \frac{D_{2k-2}(t^*)}{D_{2k-1}(t^*)} \left(\sum_{i=1}^{i=k} y_i \right)^2$$

Now, passage to the limit under the integral sign in (3.2.1), which is easily justified by application of the Lebesgue Theorem, leads to

$$E \{ Z_1^+ \dots Z_k^+ \} \approx \frac{1}{(2\pi)^{\frac{k}{2}}} k! \dots (2k-1)! \left[\prod_{1 \leq i < j \leq k} |t_j - t_i|^5 \right] \left(\frac{D_{k-1}(t^*)}{D_{2k-1}(t^*)} \right)^{\frac{1}{2}} I_k(\alpha^*)$$

where $I_k(\alpha)$, $\alpha > 0$ is

$$I_k(\alpha) = \int_{(R^+)^k} y_1 \dots y_k \exp \left[-\frac{\alpha}{2} \left(\sum_{i=1}^{i=k} y_i \right)^2 \right] dy_1 \dots dy_k = \frac{1}{\alpha^k} I_k(1)$$

and

$$\alpha^* = [(2k-1)!]^2 \frac{D_{2k-2}(t^*)}{D_{2k-1}(t^*)}$$

Replacing into (??) one gets the result with

$$J_k(t) = \frac{2! \dots (2k-2)!}{[2\pi(2k-1)!]^{2k-1}} \frac{I_k(1)}{[D_{2k-1}(t)]^{\frac{1}{2}}} \left[\frac{D_{2k-1}(t)}{D_{2k-2}(t)} \right]^k$$

This finishes the proof.

Numerical examples

We give here several examples of the computation of the distribution function of M for stationary centered Gaussian processes. The examples are listed in the following table where the covariances and the corresponding spectral densities are indicated.

process	covariance	spectral density
X_1	$\Gamma_1(t) = \exp(-t^2/2)$	$f_1(x) = (2\pi)^{-1/2} \exp(-x^2/2)$
X_2	$\Gamma_2(t) = (ch(t))^{-1}$	$f_2(x) = (2ch((\pi x)/2))^{-1}$
X_3	$\Gamma_3(t) = (3^{1/2}t)^{-1} \sin(3^{1/2}t)$	$f_3(x) = 12^{-1/2} \mathbb{1}_{\{-\sqrt{3} < x < \sqrt{3}\}}$
X_4	$\Gamma_4(t) = e^{- \sqrt{5}t } (\frac{\sqrt{5}}{3} t ^3 + 2t^2 + \sqrt{5} t + 1)$	$f_4(x) = \frac{10^4}{\sqrt{5}\pi} (5 + x^2)^{-4}$

In all cases, $\lambda_0 = \lambda_2 = 1$ to be able to compare the various results. Note that Γ_1 and Γ_3 have analytic extensions to the whole plane, so that Theorem 3.2 applies to the processes X_1 and X_3 . On the other hand, even though all spectral moments of the process X_2 are finite, Theorem 3.2 does not apply since the meromorphic extension of $\Gamma_2(\cdot)$ has poles at the points $i\pi/2 + k\pi i$, k an integer, which means that H_1 does not hold. With respect to $\Gamma_4(\cdot)$ notice that it is obtained as the convolution $\Gamma_5 * \Gamma_5 * \Gamma_5 * \Gamma_5$ where $\Gamma_5(t) := e^{-|t|}$ is the covariance of the Ornstein-Uhlenbeck process, plus a change of scale to get $\lambda_0 = \lambda_2 = 1$. The process X_4 has $\lambda_6 < \infty$ and $\lambda_8 = \infty$ and its paths are C^3 . So, for the processes X_2 and X_4 we apply Theorem 3.3 to compute $F(T, u)$. The table below, which is extracted from the work of Croquette (1999) contains the results for $T = 1$ and $T = 4$ and the values $u = -3, -2, -1, 0, 1, 2, 3$. In all cases the error is smaller than 10^{-4} except for the values indicated with a * for which it is smaller than 10^{-3} .

u	Γ_1	Γ_1	Γ_2	Γ_2	Γ_3	Γ_3	Γ_4	Γ_4
	$T = 1$	$T = 4$	$T = 1$	$T = 4$	$T = 1$	$T = 4$	$T = 1$	$T = 4$
-3	0.0001	0.0000	0.0002	0.0000	0.0001	0.0000	0.0002	0.0000
-2	0.0056	0.0000	0.0062	0.0012	0.0050	0.0000	0.0060	0.0000
-1	0.0721	0.0038	0.0753	0.0142	0.0692	0.002*	0.0747	0.004*
0	0.3473	0.0986	0.3517	0.1245	0.3437	0.081*	0.3508	0.108*
1	0.7459	0.5081	0.7475	0.5252	0.7450	0.4948	0.7472	0.517*
2	0.9558	0.8933	0.9559	0.8954	0.9557	0.8922	0.9559	0.8945
3	0.9969	0.9916	0.9969	0.9916	0.9969	0.9916	0.9969	0.9916

For $T = 1$ the results are very close to using only the first term in the Rice series in (3.1), so they are very similar between themselves, because of $\lambda_0 = \lambda_2 = 1$. For $T = 4$ three or four terms of the Rice series are needed to get the desired precision.

3.3 Processes with continuous paths

This subsection is devoted to a modification of Theorem 3.1 to include processes that do not have \mathcal{C}^∞ paths. This is done using a regularisation of the paths by convolution with a deterministic approximation of unity. For simplicity, we will limit ourselves to the case of Gaussian kernels. Other kernels can be used in a similar way.

Suppose $X = \{X_t : t \in [0, 1]\}$, is a stochastic process with continuous paths. Let ϵ be a positive real number, we define

$$X^\epsilon(t) := (\phi_\epsilon * X_{(\cdot)})(t) = \int_{-\infty}^{+\infty} \phi_\epsilon(t-s)X_s ds, \quad (37)$$

where

$$\phi_\epsilon(t) := (2\pi)^{-1/2}(\epsilon)^{-1}e^{-t^2/2\epsilon^2}, \quad t \in \mathbb{R},$$

and in (3.3) we have extended $X_{(\cdot)}$ by X_0 (respectively X_1) for $t \leq 0$ (respectively $t \geq 1$). Denote by $M^\epsilon, \nu_m^\epsilon, \dots$ the analogous to M, ν_m, \dots for the process $X^\epsilon = \{X^\epsilon(t), t \in [0, 1]\}$ instead of X . $\omega_f(\cdot)$ denotes the continuity modulus of the function f defined on $[0, 1]$.

Theorem 3.4 *With the above notations, suppose that the following conditions hold:*

- a) $p_{X^\epsilon(1/2)}(x)$ is bounded by a constant D_1 for ϵ small enough.
- b) $E(\|X\|_\infty) < \infty$.
- c) *The distribution of M has no atoms.*

Then:

(i)

$$P(M > u) = P(X_0 > u) + \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\nu_m^\epsilon}{m!} \quad (38)$$

(ii) *In formula (??) the error, when one replaces the limit by a given ϵ ($0 < \epsilon < \epsilon_0 := e^{-2}$) and the infinite sum by the m_0 partial sum is bounded by:*

$$\begin{aligned} & [32D_1E(\|X\|_\infty)]^{1/2} \Psi_{m_0+1}^{*,\epsilon} + P(|X_0 - u| < \eta) + P(u < M \leq u + \eta) + \\ & + P(\omega_X(\delta(\epsilon)) \geq \eta/2) + P(\|X\|_\infty > \frac{\sqrt{2\pi\eta}}{8\epsilon}) \end{aligned} \quad (39)$$

for each $\eta > 0$, Where

$$\delta(\epsilon) := \epsilon(2\log(1/\epsilon))^{1/2}$$

$$\Psi_m^{*,\epsilon} := \sup_{k \geq m} \left([(2k-1)!]^{1/2} \epsilon^{2k-1} \right)^{-1/2}.$$

Note: If one wishes the bound for the error in Formula (39) to be smaller than some positive number, proceed according to the following steps:

- 1) choose $\eta > 0$ so that the second and third terms are small;
- 2) with that value of η choose $\epsilon > 0$, so that the fourth and fifth terms are small;
- 3) choose m_0 large enough to make the first term small.

Proof: Consider the events

$$E_1 := \{|X_0 - u| < \eta\}, E_2 := \{u < M \leq u + \eta\},$$

$$E_3 := \{\omega_X(\delta(\epsilon)) \geq \eta/2\}, E_4 := \{\|X\|_\infty > \frac{\sqrt{2\pi}\eta}{4\epsilon}\}$$

$$E := E_1 \cup E_2 \cup E_3 \cup E_4.$$

Observe that if $\omega \notin E$ and $\epsilon < \epsilon_0$, then

$$|X^\epsilon(t) - X_t| \leq \int_{-\infty}^{+\infty} \phi_\epsilon(t-s) |X_s - X_t| ds \leq \omega_X(\delta(\epsilon)) + 2\|X\|_\infty \int_{|t-s| > \delta(\epsilon)} \phi_\epsilon(t-s) ds < \eta$$

Using this relation one gets:

$$\begin{aligned} P(M > u, X_0 \leq u) &\leq P(M > u + \eta, X_0 \leq u - \eta, E^c) + P(E) \leq \\ &\leq P(M^\epsilon > u, X^\epsilon(0) < u) + P(E) \leq P(U_u^\epsilon \geq 1, X^\epsilon(0) < u) + P(E). \end{aligned}$$

Also

$$\begin{aligned} P(U_u^\epsilon \geq 1, X^\epsilon(0) \leq u, E^c) &\leq P(U_u^\epsilon \geq 1, X^\epsilon(0) \leq u, X_0 \leq u - \eta, E^c) \leq \\ &\leq P(M > u, X_0 \leq u - \eta) \leq P(M > u, X_0 \leq u). \end{aligned}$$

Summing up:

$$\begin{aligned} P(X_0 > u) + P(U_u^\epsilon \geq 1, X^\epsilon(0) \leq u) - P(E) &\leq P(M > u) \leq \\ &\leq P(X_0 > u) + P(U_u^\epsilon \geq 1, X^\epsilon(0) \leq u) + P(E). \end{aligned}$$

To compute $P(U_u^\epsilon \geq 1, X^\epsilon(0) \leq u)$ we apply the same method as in the proof of Theorem 3.1. For that purpose, we need to show that the process X^ϵ satisfies the

conditions for an appropriate choice of the sequence $\{c_k; k = 1, 2, \dots\}$. Denoting by $H_k(s)$, the k -th Hermite polynomial, we have:

$$\begin{aligned} |X^{\epsilon(k)}(t)| &\leq \epsilon^{-(k+1)} \|X\|_\infty \int_{-\infty}^{+\infty} |\phi^{(k)}((t-s)/\epsilon)| ds = \epsilon^{-k} \|X\|_\infty \int_{-\infty}^{+\infty} |\phi^{(k)}(u)| du \\ &= \epsilon^{-k} \|X\|_\infty k! \int_{-\infty}^{+\infty} |H_k(s)| \phi(s) ds \leq \epsilon^{-k} \|X\|_\infty k! \left(\int_{-\infty}^{+\infty} (H_k(s))^2 \phi(s) ds \right)^{1/2} \\ &= \epsilon^{-k} \|X\|_\infty (k!)^{1/2}. \end{aligned}$$

So,

$$\begin{aligned} \gamma_k^\epsilon &= P(\|X^{\epsilon(2k-1)}\|_\infty \geq c_k) + \frac{D_1 c_k}{2^{2k-1} (2k-1)!} \\ &\leq \frac{((2k-1)!)^{1/2}}{\epsilon^{2k-1} c_k} E(\|X\|_\infty) + \frac{D_1 c_k}{2^{2k-1} (2k-1)!} \end{aligned}$$

Choosing

$$c_k := \left[\frac{((2k-1)!)^{3/2} E(\|X\|_\infty)}{(\epsilon/2)^{2k-1} D_1} \right]^{1/2}.$$

we obtain

$$\gamma_k^\epsilon \leq 2^{-k} \left[\frac{8D_1 E(\|X\|_\infty)}{\epsilon^{2k-1} ((2k-1)!)^{1/2}} \right]^{1/2}.$$

Hence,

$$\gamma_m^{\epsilon*} = \sup_{k \geq m} (2^{k+1} \gamma_k^\epsilon) \leq [32D_1 E(\|X\|_\infty)]^{1/2} \Psi_m^{*,\epsilon}.$$

The remaining follows as in the proof of Theorem 3.1.

Remarks and examples

A) Conditions a), b) and c) in Theorem 3.4 are usually non trivial to check and the a priori estimation of the error can be a hard problem. Moreover, when this can be actually done, the validity of Rice formulae and the feasibility of the method still remains a problem if one is willing to use Theorem 3.4 as a tool for numerical computation. For a given error, smaller $\epsilon > 0$ implies larger m_0 and the usefulness of Theorem 3.4 for numerical applications is limited. The bound in (39) shows that a priori we require at least $m_0 \simeq (1/2)\epsilon^{-2}$ terms in the sum as $\epsilon \rightarrow 0$.

B) Let X be a Gaussian process with continuous paths and

$$m(t) := E(X_t) \quad ; \quad \sigma^2(t) := Var(X_t) > 0$$

be the (continuous) mean and variance of X_t . Condition a) in Theorem 3.4 follows easily together with bounds on D_1 and $P(E_1)$.

Condition b) is well-known from classical results on Gaussian processes (Fernique, (1974)) that also give bounds on $E(\|X\|_\infty)$ and on the tail probability $P(\|X\|_\infty > a)$ from additional information on the process X (see Adler, (1990), Borell, (1975), Ledoux and Talagrand, (1991), Ledoux, (1996), Sun (1993) and references therein). This implies bounds on $P(E_4)$.

Condition c) is Ylvisaker's theorem (1968). A priori bounds on $P(E_2)$ follow from bounds on the density of the distribution of the random variable M , resulting from additional conditions on the Gaussian process X (see Diebolt and Posse (1996) and references therein, Weber (1985), Azaïs and Wschebor (1999, 2000).

$P(E_3)$ can be bounded using the classical methods to study the continuity modulus of a stochastic process, for example, if the incremental variance $V_t(h) = E((X_{t+h} - X_t)^2)$ is given (see for example Cramér and Leadbetter (1967)).

C) Theorem 3.4 can be applied to one-dimensional diffusions satisfying certain assumptions. Let $\{X_t : t \geq 0\}$ be the strong solution of the stochastic differential equation

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt \quad ; \quad X_0 = x_0,$$

where $\{W_t : t \geq 0\}$ stands for the standard Brownian motion, $\sigma, b : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and $\frac{\partial \sigma}{\partial x}, \frac{\partial b}{\partial x}$ are continuous and bounded and $x_0 \in \mathbb{R}$. We also assume that

$$\sigma(t, x) \geq \sigma_0 > 0 \quad , \quad t \in \mathbb{R}^+, x \in \mathbb{R}.$$

The methods employed in Azaïs (1989) or Nualart and Wschebor (1991) permit to prove that $p_{X^\varepsilon(t)}$ exists and is a bounded function for $t \in [\delta, 1]$ for each $\delta > 0, 0 < \varepsilon < \varepsilon_0(\delta)$. Condition b) is standard and well known. As for condition c), it follows as in Nualart and Vives (1988) using stochastic calculus of variations.

Hence, Theorem 3.4 can be used to obtain formula (??) for $P(M_\delta > u)$, $M_\delta := \max_{\delta \leq t \leq 1} X_t$ and bounds having the form (39) for the error. Adding an elementary bound on the local oscillation $P(\max_{0 \leq t \leq \delta} |X_t - x_0| \leq \eta)$, one is able to get $P(M > u)$ with a controlled error. On the other hand, an obstacle to have an actual numerical computation for $P(M > u)$ is the lack of a good description of the joint densities of $X^\varepsilon(t), X^{\varepsilon'}(t)$ at the k -tuple (t_1, \dots, t_k) to be used in Rice formulae. This problem does not have yet a satisfactory solution.

D) Even though polygonal approximations are not differentiable, they can be used instead of smoothing in Theorem 3.4. In this case, the advantage is that the number of terms in the Rice series is finite. We will not pursue the subject here.

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